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A q-difference version of the ϵ-algorithm

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Abstract
In this paper, a q-difference version of the ϵ-algorithm is proposed. By using determinant identities the solutions of an initial value problem thus arisen can be expressed as ratios of Hankel determinants. It is shown that in numerical analysis this algorithm can be used to compute the approximation $\lim_{t\to\infty} f(t)$, and in the field of integrable systems it could be viewed as the q-difference version of the modified Toda molecule equation.

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1. Introduction

Recently, it has been shown that integrable systems have close connections with numerical algorithms. This notion brings a fresh look in the research of both fields and much interesting work has been done subsequently. For example, one step of the QR algorithm is equivalent to the time evolution of the finite nonperiodic Toda lattice [1]. The ϵ-algorithm is nothing but the fully discrete potential KdV equation [2]. The η-algorithm and ρ-algorithm are considered to be the fully discrete KdV and fully discrete cylindrical KdV equations, respectively [3]. The discrete Lotka–Volterra system has applications in numerical algorithms for computing singular values [4–6].

In this short paper, we construct a q-difference version of the ϵ-algorithm for convergence acceleration. Let $\{S_n\}$ be a sequence converging to some limit S. Sometimes the convergence of the sequence is slow, so one turns to construct convergence acceleration algorithms which transform the original sequence to a new one. We say that the transformation $T : \{S_n\} \to \{T_n\}$ accelerates the convergence of the sequence $\{S_n\}$ if

$$\lim_{t\to\infty} \frac{T_n - S}{S_n - S} = 0.$$

In the literature, many convergence acceleration transformations for sequences have been constructed [7]. One of them is the famous Shank's transformation [8], which is the general case of the well-known Aitken's Δ^2 process. In 1956, Wynn derived a recursive procedure to compute the new sequences that Shank's transformation produces. This is the famous ϵ-algorithm [9] and is given by

$$\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + \frac{1}{\epsilon_k^{(n+1)} - \epsilon_k^{(n)}}, \tag{1}$$

with initial conditions

$$\epsilon_{-1}^{(n)} = 0, \qquad \epsilon_0^{(n)} = S_n.$$

Wynn obtained in [10] again the confluent form of the ϵ-algorithm:

$$\epsilon_{k+1}(t) = \epsilon_{k-1}(t) + \frac{1}{\epsilon_k'(t)}, \tag{2}$$

with initial conditions

$$\epsilon_{-1}(t) = 0, \qquad \epsilon_0(t) = f(t).$$

The purpose of this algorithm is to compute the approximation $\lim_{t\to\infty} f(t)$. Setting $N_k(t) = \epsilon_k'(t)\epsilon_{k+1}'(t)$, this algorithm reduces to the Bäcklund transformation of the discrete Toda molecule equation [11]

$$N_k'(t) = N_k(t)[N_{k-1}(t) - N_{k+1}(t)],$$

which is also called the Lotka–Volterra lattice.

In this paper, first we propose the q-difference version of the ϵ-algorithm. Then we study an initial value problem with this algorithm and also the kernel and integrability of this transformation.

This paper is organized as follows. In section 2, we will give the q-difference form of the ϵ-algorithm. In section 3, we derive the solution to an initial value problem with this algorithm. In section 4, we study the kernel of the q-difference ϵ-algorithm and the relationship to integrable systems. In section 5, examples of application of this algorithm are presented. Section 6 is devoted to conclusions.

2. Derivation of the q-difference ϵ-algorithm

In this section, we show how the q-difference ϵ-algorithm is derived and also give its property. Consider the ϵ-algorithm (1). Set $t = (q^\alpha)^n x_0$, replace $\epsilon_k^{(n)}$ by $\epsilon_{2k}(t)$, and also $\epsilon_{2k+1}^{(n)}$ by $\epsilon_{2k+1}(t)/((q-1)t)$, where $q > 1$, $x_0 > 0$, and $\alpha > 0$ are constants. (These assumptions will be kept throughout this paper.) Under the assumptions, $t \to \infty$ when $n \to \infty$. Then we get the q-difference ϵ-algorithm

$$\epsilon_{k+1}(t) = \epsilon_{k-1}(q^\alpha t) + \frac{1}{\delta_{q^\alpha}\epsilon_k(t)}, \tag{3}$$

where the q-difference operator δ_{q^α} is defined by [12]

$$\delta_{q^\alpha} f(t) = \frac{f(q^\alpha t) - f(t)}{(q-1)t}. \tag{4}$$

Note that when $\alpha = 1$ and $q \to 1$, the above algorithm reduces to the confluent ϵ-algorithm (2). Furthermore, this algorithm has the following property.

Theorem 1. *Let* $\{\epsilon_k(t)\}$ *be constructed by relation (3) from the initial condition*

$$\epsilon_{-1}(t) = 0, \qquad \epsilon_0(t) = f(t),$$

and $\{\bar{\epsilon}_k(t)\}$ *constructed by the same relation from the initial condition*

$$\bar{\epsilon}_{-1}(t) = 0, \qquad \bar{\epsilon}_0(t) = af(t) + b,$$

then

$$\bar{\epsilon}_{2k}(t) = a\epsilon_{2k}(t) + b, \qquad \bar{\epsilon}_{2k+1}(t) = \frac{\epsilon_{2k+1}(t)}{a},$$

where $a \neq 0$ *and* b *are constants.*

Proof. The above results can easily be proved by induction from the recursion relation (3). □

3. Determinant solution of an initial value problem with the q-difference ϵ-algorithm

Define a sequence of Hankel determinants

$$H_k^{(n)}(t) \equiv \begin{vmatrix} \delta_{q^\alpha}^n f(t) & \delta_{q^\alpha}^{n+1} f(t) & \cdots & \delta_{q^\alpha}^{n+k-1} f(t) \\ \delta_{q^\alpha}^{n+1} f(t) & \delta_{q^\alpha}^{n+2} f(t) & \cdots & \delta_{q^\alpha}^{n+k} f(t) \\ \vdots & \vdots & & \vdots \\ \delta_{q^\alpha}^{n+k-1} f(t) & \delta_{q^\alpha}^{n+k} f(t) & \cdots & \delta_{q^\alpha}^{n+2k-2} f(t) \end{vmatrix}, \qquad k = 1, 2, \ldots,$$

$$H_{-1}^{(n)} \equiv 0, \qquad H_0^{(n)} \equiv 1, \qquad n = 1, 2, \ldots.$$

Then we have the following result.

Theorem 2. *Given the initial values of the q-difference algorithm*

$$\epsilon_{-1}(t) = 0, \qquad \epsilon_0(t) = f(t), \tag{5}$$

then $\{\epsilon_k(t)\}$ *constructed from relation (3) have the explicit formulae*

$$\epsilon_{2k-1}(t) = \frac{H_{k-1}^{(3)}(t)}{H_k^{(1)}(t)}, \qquad \epsilon_{2k}(t) = \frac{H_{k+1}^{(0)}(t)}{H_k^{(2)}(t)}.$$

Proof. First, we prove the following bilinear equations:

$$H_k^{(n+2)}(q^\alpha t)\delta_{q^\alpha} H_{k+1}^{(n)}(t) - H_{k+1}^{(n)}(q^\alpha t)\delta_q^\alpha H_k^{(n+2)}(t) = H_k^{(n+1)}(q^\alpha t) H_{k+1}^{(n+1)}(t), \tag{6}$$

$$H_{k+1}^{(n)}(t) H_{k-1}^{(n+2)}(q^\alpha t) = H_k^{(n+2)}(t) H_k^n(q^\alpha t) - H_k^{(n+1)}(t) H_k^{(n+1)}(q^\alpha t). \tag{7}$$

Let D be some determinant, $\begin{bmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{bmatrix}$ denotes the determinant with the $i_1 < i_2 < \cdots < i_n$th rows and $j_1 < j_2 < \cdots < j_n$th columns removed from D. We start with defining

$$D \equiv \begin{vmatrix} \delta_{q^\alpha}^n f(q^\alpha t) & \cdots & \delta_{q^\alpha}^{n+k} f(q^\alpha t) & 0 \\ \vdots & & \vdots & \vdots \\ \delta_{q^\alpha}^{n+k-1} f(q^\alpha t) & \cdots & \delta_{q^\alpha}^{n+2k-1} f(q^\alpha t) & 0 \\ \delta_{q^\alpha}^{n+k} f(q^\alpha t) & \cdots & \delta_{q^\alpha}^{n+2k} f(q^\alpha t) & 1 \\ \delta_{q^\alpha}^{n+k+1} f(t) & \cdots & \delta_{q^\alpha}^{n+2k+1} f(t) & 0 \end{vmatrix}.$$

From the definition of $\delta_{q^\alpha} f(t)$ in (4), we have

$$D = -\delta_{q^\alpha} H_{k+1}^{(n)}(t).$$

In addition, we can obtain the relations

$$D\begin{bmatrix}1 & k+2\\ 1 & k+2\end{bmatrix} = H_k^{(n+2)}(q^\alpha t), \qquad D\begin{bmatrix}k+2\\ k+2\end{bmatrix} = H_{k+1}^{(n)}(q^\alpha t),$$

$$D\begin{bmatrix}1\\ 1\end{bmatrix} = -\delta_{q^\alpha} H_k^{(n+2)}(t), \qquad D\begin{bmatrix}1\\ k+2\end{bmatrix} = H_{k+1}^{(n+1)}(t),$$

$$D\begin{bmatrix}k+2\\ 1\end{bmatrix} = H_k^{(n+1)}(q^\alpha t).$$

From the above results, we see that the bilinear equation (6) is equivalent to the Jacobi identity

$$DD\begin{bmatrix}1 & k+2\\ 1 & k+2\end{bmatrix} = D\begin{bmatrix}1\\ 1\end{bmatrix} D\begin{bmatrix}k+2\\ k+2\end{bmatrix} - D\begin{bmatrix}1\\ k+2\end{bmatrix} D\begin{bmatrix}k+2\\ 1\end{bmatrix}.$$

□

Next we prove the other determinant identity (7). Define

$$\bar{D} \equiv \begin{vmatrix} \delta_{q^\alpha}^{n} f(q^\alpha t) & \cdots & \delta_{q^\alpha}^{n+k-1} f(q^\alpha t) & \delta_{q^\alpha}^{n+k} f(q^\alpha t)\\ \vdots & & \vdots & \vdots\\ \delta_{q^\alpha}^{n+k-1} f(q^\alpha t) & \cdots & \delta_{q^\alpha}^{n+2k-2} f(q^\alpha t) & \delta_{q^\alpha}^{n+2k-1} f(q^\alpha t)\\ \delta_{q^\alpha}^{n+k} f(t) & \cdots & \delta_{q^\alpha}^{n+2k-1} f(t) & \delta_{q^\alpha}^{n+2k} f(t) \end{vmatrix},$$

then we have the relations

$$\bar{D} = H_{k+1}^{(n)}(t), \qquad \bar{D}\begin{bmatrix}1 & k+1\\ 1 & k+1\end{bmatrix} = H_{k-1}^{(n+2)}(q^\alpha t),$$

$$\bar{D}\begin{bmatrix}1\\ 1\end{bmatrix} = H_k^{(n+2)}(t), \qquad \bar{D}\begin{bmatrix}k+1\\ k+1\end{bmatrix} = H_k^{(n)}(q^\alpha t),$$

$$\bar{D}\begin{bmatrix}1\\ k+1\end{bmatrix} = H_k^{(n+1)}(t), \qquad \bar{D}\begin{bmatrix}k+1\\ 1\end{bmatrix} = H_k^{(n+1)}(q^\alpha t),$$

from which we see that the bilinear equation (7) is nothing but the Jacobi identity

$$\bar{D}\bar{D}\begin{bmatrix}1 & k+1\\ 1 & k+1\end{bmatrix} = \bar{D}\begin{bmatrix}1\\ 1\end{bmatrix} \bar{D}\begin{bmatrix}k+1\\ k+1\end{bmatrix} - \bar{D}\begin{bmatrix}1\\ k+1\end{bmatrix} \bar{D}\begin{bmatrix}k+1\\ 1\end{bmatrix}.$$

Now we consider equation (3). When k is odd, by the bilinear identity (7) we have

$$\begin{aligned} \epsilon_{2k}(t) - \epsilon_{2k-2}(q^\alpha t) &= \frac{H_{k+1}^{(0)}(t)}{H_k^{(2)}(t)} - \frac{H_k^{(0)}(q^\alpha t)}{H_{k-1}^{(2)}(q^\alpha t)}\\ &= \frac{H_{k+1}^{(0)}(t)H_{k-1}^{(2)}(q^\alpha t) - H_k^{(2)}(t)H_k^{(0)}(q^\alpha t)}{H_k^{(2)}(t)H_{k-1}^{(2)}(q^\alpha t)}\\ &= -\frac{H_k^{(1)}(t)H_k^{(1)}(q^\alpha t)}{H_k^{(2)}(t)H_{k-1}^{(2)}(q^\alpha t)}. \end{aligned}$$

On the other hand, by identity (6) we have

$$\begin{aligned}\delta_{q^\alpha}\epsilon_{2k-1}(t) &= \delta_{q^\alpha}\frac{H_{k-1}^{(3)}(t)}{H_k^{(1)}(t)}\\ &= \frac{H_k^{(1)}(q^\alpha t)\delta_{q^\alpha}H_{k-1}^{(3)}(t) - H_{k-1}^{(3)}(q^\alpha t)\delta_{q^\alpha}H_k^{(1)}(t)}{H_k^{(1)}(t)H_k^{(1)}(q^\alpha t)}\\ &= -\frac{H_k^{(2)}(t)H_{k-1}^{(2)}(q^\alpha t)}{H_k^{(1)}(t)H_k^{(1)}(q^\alpha t)}.\end{aligned}$$

Therefore we have proved equation (3) for the odd subscripts. Equation (3) for the even subscripts can be proved in a similar way.

4. Kernel of the q-difference ϵ-algorithm and the relationship to integrable systems

In this section, we give the kernel of this transformation and show the relationship to integrable systems.

From the expression of $\epsilon_{2k}(t)$ in theorem 2, we derive the kernel of the q-difference ϵ-algorithm.

Theorem 3. *Let $\{\epsilon_k(t)\}$ be derived by the q-difference ϵ-algorithm with initial condition (5). Assume T is a constant. Then a necessary and sufficient condition that $\epsilon_{2k}(t) = S$ for all $t \geqslant T$ is given by*

$$f(t) = S + a_1\delta_{q^\alpha}f(t) + a_2\delta_{q^\alpha}^2 f(t) + \cdots + a_k\delta_{q^\alpha}^k f(t), \qquad (8)$$

where $a_i, i = 1, \ldots, k$ are constants, or equivalently,

$$f(t) = S + b_1\delta_{q^\alpha}f(t) + b_2\delta_{q^\alpha}f(q^\alpha t) + \cdots + b_k\delta_{q^\alpha}f((q^\alpha)^{k-1}t), \qquad (9)$$

where $b_i, i = 1, \ldots, k$ are functions of t which can be obtained from the equivalent expressions of $f(t)$ as in (8) and (9).

Proof. The sufficient condition is obvious so that we just need to prove the necessary condition. If $\forall\, t \geqslant T, \epsilon_{2k}(t) = S$, by theorem 2 it is equivalent to

$$H_{k+1}^{(0)}(t) = SH_k^{(2)}(t).$$

After rearranging, we arrive at

$$H_{k+1}^{(0)}(t) - SH_k^{(2)}(t) = \begin{vmatrix} f(t)-S & \delta_{q^\alpha}f(t) & \cdots & \delta_{q^\alpha}^k f(t)\\ \delta_{q^\alpha}f(t) & \delta_{q^\alpha}^2 f(t) & \cdots & \delta_{q^\alpha}^{k+1}f(t)\\ \vdots & \vdots & & \vdots\\ \delta_{q^\alpha}^k f(t) & \delta_{q^\alpha}^{k+1}f(t) & \cdots & \delta_{q^\alpha}^{2k}f(t)\end{vmatrix} = 0,$$

which is equal to

$$\begin{cases} c_0(f(t)-S) + c_1\delta_{q^\alpha}f(t) + \cdots + c_k\delta_{q^\alpha}^k f(t) = 0\\ c_0\delta_{q^\alpha}f(t) + c_1\delta_{q^\alpha}^2 f(t) + \cdots + c_k\delta_{q^\alpha}^{k+1}f(t) = 0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ c_0\delta_{q^\alpha}^k f(t) + c_1\delta_{q^\alpha}^{k+1}f(t) + \cdots + c_k\delta_{q^\alpha}^{2k}f(t) = 0\end{cases}$$

where $c_i, i = 0, 1, \ldots, k$ are some constants. This equation is equivalent to equation (8). Hence the proof is completed. □

Next we consider the relationship of the q-difference ϵ-algorithm with integrable systems. Setting $u_k(t) = \delta_{q^\alpha}\epsilon_k(t)$, from equation (3) we have

$$\delta_{q^\alpha}u_k(t) = u_k(t)u_k(q^\alpha t)[u_{k-1}(q^\alpha t) - u_{k+1}(t)]. \tag{10}$$

On the other hand, considering the confluent ϵ-algorithm (2), set $v_k(t) = \epsilon_k'(t)$, this algorithm reduces to

$$v_k'(t) = v_k^2(t)[v_{k-1}(t) - v_{k+1}(t)], \tag{11}$$

which is the modified Toda molecule equation. So we say that equation (10) is the q-difference version of the modified Toda molecule equation (11). From theorems 2 and 3, $\{u_k(t)\}_k$ can be expressed as

$$u_{2k-1}(t) = -\frac{H_{k-1}^{(2)}(q^\alpha t)H_k^{(2)}(t)}{H_k^{(1)}(q^\alpha t)H_k^{(1)}(t)},$$

$$u_{2k}(t) = \frac{H_k^{(1)}(q^\alpha t)H_{k+1}^{(1)}(t)}{H_k^{(2)}(q^\alpha t)H_k^{(2)}(t)}.$$

5. Convergence acceleration examples

In this section, we apply the q-difference ϵ-algorithm to accelerate the convergence of function.

Example 1. Set $f(t) = \frac{1}{e_{q^\alpha}(t)}$, where $e_{q^\alpha}(t)$ is defined by [13]

$$e_{q^\alpha}(t) = \sum_{k=0}^{\infty}\frac{t^k}{[k]!},$$

and

$$[n] = \frac{1-(q^\alpha)^n}{1-q}, \qquad [n]! = [1][2]\cdots[n], \qquad [0]! = 1.$$

From the recurrence equation (3) with initial conditions

$$\epsilon_{-1}(t) = 0, \qquad \epsilon_0(t) = f(t),$$

it can be computed by induction that

$$\epsilon_{2k}(t) = \left(1 - \frac{1}{q^\alpha}\right)\left(1 - \frac{1}{q^{2\alpha}}\right)\cdots\left(1 - \frac{1}{q^{k\alpha}}\right)\frac{1}{e_{q^\alpha}(q^{k\alpha}t)}$$

$$\epsilon_{2k+1}(t) = -\frac{q^\alpha q^{2\alpha}\cdots q^{k\alpha}}{(q^\alpha - 1)(q^{2\alpha}-1)\cdots(q^{k\alpha}-1)}e_{q^\alpha}(q^{(k+1)\alpha}t).$$

From above expressions, we see that

$$\lim_{t\to\infty}\frac{\epsilon_{2k}(t)}{f(t)} = 0,$$

$$\frac{\epsilon_{2k+2}(t)}{\epsilon_{2k}(t)} = \left(1 - \frac{1}{q^{(k+1)\alpha}}\right)\frac{e_{q^\alpha}(q^{k\alpha}t)}{e_{q^\alpha}(q^{(k+1)\alpha}t)},$$

$$\lim_{t\to\infty}\frac{\epsilon_{2k+2}(t)}{\epsilon_{2k}(t)} = 0.$$

For $t > 0$

$$\lim_{k\to\infty} \frac{\epsilon_{2k+2}(t)}{\epsilon_{2k}(t)} = 0.$$

From the above results, we see that $\epsilon_{2k}(t)$ converges faster than $f(t)$ and also $\epsilon_{2k+2}(t)$ converges faster than $\epsilon_{2k}(t)$.

Example 2. Set $f(t) = t/e^t$. With different choices of q and α, the corresponding numerical results are presented.

Case 1. Let $q = 2, \alpha = 2$.

t	$f(t)$	$\epsilon_2(t)$	$\epsilon_4(t)$
1	0.367 879 4412	0.053 733 309 34	$1.266\,017\,472 \times 10^{-6}$
2	0.270 670 5664	0.002 011 091 840	$2.849\,437\,249 \times 10^{-13}$
3	0.149 361 2051	0.000 055 295 635 01	$4.809\,928\,781 \times 10^{-20}$
4	0.073 262 555 56	0.000 001 350 419 331	$7.217\,149\,007 \times 10^{-27}$

Case 2. Let $q = 1.2, \alpha = 2$.

t	$f(t)$	$\epsilon_2(t)$	$\epsilon_4(t)$
1	0.367 879 4412	0.392 325 1632	0.016 821 489
2	0.270 670 5664	0.010 471 0148	0.007 017 034 12
3	0.149 361 2051	0.009 960 654 55	0.001 156 312 864
4	0.073 262 555 56	0.004 148 954 51	0.000 259 707 9314

Case 3. Let $q = 2, \alpha = 1$.

t	$f(t)$	$\epsilon_2(t)$	$\epsilon_4(t)$
1	0.367 879 4412	6.688 107 100	0.021 627 323 72
2	0.270 670 5664	0.030 291 444 60	$9.776\,275\,778 \times 10^{-4}$
3	0.149 361 2051	0.007 042 314 565	$2.754\,531\,975 \times 10^{-5}$
4	0.073 262 555 56	0.001 316 780 228	$6.748\,704\,838 \times 10^{-7}$

Example 3. Set $f(t) = 1 - t\sin(1/t)$. Similarly, the numerical results with different choices of q and α are presented.

Case 1. Let $q = 2, \alpha = 3$.

t	$f(t)$	$\epsilon_2(t)$	$\epsilon_4(t)$
5	0.006 653 3460	$9.132\,145\,490 \times 10^{-5}$	$1.404\,879\,816 \times 10^{-6}$
10	0.001 665 8335	$2.283\,100\,115 \times 10^{-5}$	$3.512\,961\,827 \times 10^{-7}$
20	0.000 416 6146	$5.707\,757\,162 \times 10^{-6}$	$8.787\,752\,584 \times 10^{-8}$
40	0.000 104 1636	$1.426\,833\,054 \times 10^{-6}$	$2.184\,016\,611 \times 10^{-8}$

Case 2. Let $q = 1.5, \alpha = 3$.

t	$f(t)$	$\epsilon_2(t)$	$\epsilon_4(t)$
5	0.006 653 3460	$4.227\,805\,414 \times 10^{-4}$	$3.412\,685\,758 \times 10^{-5}$
10	0.001 665 8335	$1.057\,101\,053 \times 10^{-4}$	$8.531\,769\,446 \times 10^{-6}$
20	0.000 416 6146	$2.642\,887\,902 \times 10^{-5}$	$2.133\,021\,198 \times 10^{-6}$
40	0.000 104 1636	$6.607\,220\,604 \times 10^{-6}$	$5.332\,189\,098 \times 10^{-7}$

Case 3. Let $q = 2, \alpha = 1$.

t	$f(t)$	$\epsilon_2(t)$	$\epsilon_4(t)$
5	0.006 653 3460	$9.518\,026\,350 \times 10^{-4}$	$1.904\,448\,703 \times 10^{-4}$
10	0.001 665 8335	$2.380\,592\,242 \times 10^{-4}$	$4.761\,721\,803 \times 10^{-5}$
15	0.000 416 6146	$5.952\,170\,194 \times 10^{-5}$	$1.190\,466\,159 \times 10^{-5}$
40	0.000 104 1636	$1.488\,085\,714 \times 10^{-5}$	$2.976\,123\,519 \times 10^{-6}$

From the above results, we see that the convergence acceleration speed becomes larger when q or α increases.

6. Conclusions

In this paper, the q-difference form of the ϵ-algorithm is constructed. The kernel of the transformation, its recursive equation, and the corresponding solutions are given. In the context of soliton theory, the q-difference version of the ϵ-algorithm can be considered as the q-difference modified Toda molecule equation. From the viewpoint of numerical analysis, this algorithm can be applied to compute $\lim_{t\to\infty} f(t)$. From the examples of the algorithm's application we see that q and α play an important role in the convergence acceleration speed.

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